

An Integral Method for Calculating Heat and Mass Transfer in Laminar Boundary Layers

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An integral method previously used successfully in several kinds of boundary layer problems has been extended to treat simultaneous heat and mass transfer in a binary-mixture laminar boundary layer when the pressure is uniform. The principal results are two pairs of dual integral relations arising from solutions to the integral concentration and energy equations. One pair connects the surface mass transfer rate and surface concentration of injected gas; the other relates surface temperature and heat transfer rate in the presence of mass transfer. Only the cases of helium and air injection into an undissociated air stream are discussed in detail, but the method can be applied to problems involving other gases. The approximate results agree quite well with some numerical solutions and with recent experimental results for which no numerical solutions are available.

Introduction

MASS-transfer cooling is one possible method of reducing the heat flow to surfaces immersed in high-energy gaseous streams. In practice, only part of the surface will be cooled directly in this manner; a gas, perhaps different from the freestream gas, then is injected through a porous region near the leading edge. An important problem is determination of the extent to which the surface downstream of the injection region is cooled. For a realistic description of the problem, one therefore should consider a binary-mixture boundary layer with nonuniform heat flow, mass transfer rate, temperature, and pressure at the surface. Fortunately, useful results still are obtained, for a flat plate and a cone in supersonic flow, if the last condition is relaxed. The subject treated here is an approximate integral method for computing relationships between the first three quantities when the pressure is uniform; a fourth important variable, the wall concentration of injected gas, also will appear.

The equations governing laminar boundary layer flows can be solved "exactly" only for special, although useful, conditions. These results are usually similarity solutions; the appropriate partial differential equations are reduced to nonlinear ordinary differential equations, which then are solved numerically. In such cases, the surface boundary conditions and freestream variations are restricted to specific forms. Relevant examples of this sort of analysis are given in Refs. 1 and 2 for helium injected through isothermal flat surfaces (flat plate and "wedge" solutions). To meet the similarity conditions, the surface mass transfer rate must vary inversely with some power of distance from the leading edge. Corresponding results for the case of air injected through a flat plate in an air stream have been presented in Refs. 1 and 3.

More recently, some results⁴⁻⁶ have been obtained for "nonsimilar" problems associated with other boundary conditions. Although these are accurately known solutions, an obvious disadvantage is that, even if the same gases are considered, separate and generally lengthy calculations must be carried out for each new problem, i.e., whenever the form of the boundary conditions is altered. It clearly is desirable to construct approximate methods, preferably based on

available numerical solutions, to handle the kinds of boundary conditions encountered in practical situations.

Similar circumstances of course have arisen previously in the development of laminar boundary layer theory. Approximate methods based on von Kármán's integral momentum equation first were developed for low speed, constant property boundary layer flows over curved surfaces. Although for many years the usual approach, first used by Pohlhausen, was to approximate the boundary layer velocity profile (e.g., by a polynomial of some kind), the most efficient calculation seems to be Thwaites' method,⁷ which, although, still based on von Kármán's equation, makes explicit use of the exact solutions. One then can calculate the boundary layer momentum thickness with a minimum of effort and with good results, especially for favorable pressure gradients. Although identical results may be obtained by a method due to Eckert⁸ and by several other approaches (see Ref. 9 for a brief survey), the viewpoint taken by Thwaites is distinctly different. Many of the investigations carried out for low speed flow have been extended with varying success to compressible flow.

The development of integral methods to compute heat transfer rates for variable wall temperatures and pressure has been similar to that of methods relating to the momentum boundary layer. Approximations to the various profiles, following Pohlhausen's early ideas, have been attempted for both compressible¹⁰⁻¹² and incompressible flow. An important comparison of several other integral methods has been carried out by Hill¹³ for the constant property flow problem resulting from application of the transformation due to Stewartson¹⁶ and Illingworth.¹⁷ Most significant to the work here is a method, developed by Hill,¹³⁻¹⁵ based on an approximate analysis, by Rott and Crabtree,⁹ of the boundary layer on a yawed cylinder. In spirit, these studies are similar to Thwaites' work, since known solutions are used explicitly. Furthermore, the momentum thickness, found most easily from Thwaites' formula, is involved in calculations of other quantities; for example, the heat transfer to a variable temperature curved surface is related to the momentum thickness by Hill's solution of the energy equation.

There are several investigations¹⁸⁻²² dealing with a problem related to that studied here. They treat approximate solutions to the equation describing the concentration of a species in a chemically active flow reacting with a surface; heat transfer has not been treated extensively by them. The results rely in large measure on analogous treatments, such as Lighthill's analysis²³ of heat transfer in incompressible flow; it will be seen that such analogies are not strictly valid

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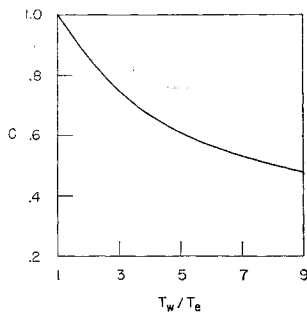


Fig. 1 Variation of proportionality constant C with surface temperature

when there is surface mass transfer. It appears that the procedure discussed below may be applicable, after suitable modification, to the problem of reacting surfaces. Although some work has been done^{10, 24, 25} on approximate methods for the problem of air injection into an air boundary layer, the accuracy of these results seems in doubt, and the analyses are restricted to rather special cases. So far as the author is aware, except for a discussion in Ref. 1, Covert's work²⁶ is the only approximate treatment of the binary mixture boundary layer with helium injection; subsequent calculations⁴ support his conclusion that his results, which are quite difficult to apply, are accurate only for low mass transfer rates.

There is therefore a need for approximate methods taking account of mass transfer through the boundary surface. Certainly the obvious approach is to attempt some extension of existing analyses. Of the integral methods used for problems not involving mass transfer, there appear to be two general classes: one comprises those investigations in which the various boundary layer profiles, appearing in integrals over the layer, are approximated in some manner, and the other includes studies in which the integrals are used directly as dependent variables, following Thwaites' work. Owing in part to the characteristic that they make full use of the numerical (similarity) solutions, the second class of methods seems to give more accurate results and is applied more easily. Consequently, the present work follows a course similar to that traced by Thwaites and Hill. It will become apparent later that the problem treated here is (roughly) analogous to the problem of heat transfer to an impermeable surface with variable temperature and freestream pressure. Since the numerical results readily are available, the results obtained are for air or helium injection into an undissociated air stream. Evidently, the method should apply equally well to problems involving other gases. It should be noted that the discussion is for flat plate flow, but, by applying Mangler's transformation, one can use the formulas for flow past a cone in supersonic flow.

The results cited contain several important ideas that will be used in the mass transfer problem. The task of representing profiles in a binary mixture boundary layer becomes monumental, since the concentration of species is an added variable; such a procedure also ignores the existence of numerical solutions except as a means to check. It is clearly advisable to avoid such approximations by using the various thicknesses, such as the momentum thickness θ and convection thickness δ_c , as dependent variables. Thus one must use numerical solutions for special boundary conditions to evaluate certain functions that then are assumed to remain unchanged when other boundary conditions apply. In effect, the assertion is that, approximately, there are certain "universal" properties of the boundary layer flow; the problem is to extract this information from known results and in such a form as to be more generally useful. Particularly, the method of Rott and Crabtree enables one to determine approximately the connection between an integral thickness such as δ_c and the right-hand side of the associated first-order equation. This gives a way to generalize special solutions, and thus one can analyze problems involving rather difficult boundary conditions. Involved in this scheme are certain

assumptions that can be justified only by comparing the final results with numerical solutions or with measurements. Somewhat more extensive accounts of the work covered here may be found in Refs. 27 and 28.

Momentum Thickness with Surface Mass Transfer

Eventually both the integral concentration and energy equations will be solved, but the first step is to construct a formula for the momentum thickness. The viewpoint from which the following calculation proceeds is like that of Thwaites. The differential momentum equation has the usual form; for uniform pressure, the equation for the momentum thickness is

$$\frac{d\theta}{dx} = \frac{1}{\rho_e u_e^2} \left(\mu \frac{\partial u}{\partial y} \right)_w + \frac{(\rho v)_w}{\rho_e u_e} \quad (1)$$

and in compressible flows

$$\theta = \int_0^\infty \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e} \right) dy \quad (2)$$

Equation (1) may be written in normalized form:

$$\frac{\rho_e u_e}{C \mu_e} \frac{d\theta^2}{dx} = 2 \frac{\mu_w}{C \mu_e} \left[\frac{\theta}{u_e} \left(\frac{\partial u}{\partial y} \right)_w + \frac{\theta (\rho v)_w}{\mu_w} \right] \quad (3)$$

where C is the Chapman-Rubens constant appearing in the analysis of binary mixture flows¹ through the relation

$$\rho \mu / \rho_e \mu_e = C \lambda \quad (4)$$

When the surface temperature varies, C depends on x , although an average value may be sufficiently accurate for some purposes; C is computed in such a way that the viscosity coefficient evaluated at the wall obeys Sutherland's formula appropriately modified to allow for variable concentration (see Ref. 1). The dependence on concentration is contained in λ , whereas C is a function of T_w/T_e shown in Fig. 1.

It is not unreasonable to assume that the velocity profiles, and hence θ , belong to a one-parameter family, similar to a postulate invoked⁷ when $(\rho v)_w = 0$, $dp/dx \neq 0$. Since, in practical cases, $(\rho v)_w$ almost certainly will be prescribed, it is desirable to relate the right-hand side of Eq. (4) to a dimensionless form of this quantity. The natural choice, f_w , arises in the analysis of Ref. 1:

$$f_w = -2(R_x/C)^{1/2} [(\rho v)_w / \rho_e u_e] \quad (5)$$

The meaning of f will become apparent later; for the present, it is sufficient to remark that f_w is independent of x for the similarity conditions and otherwise will be a function of x as well as the level of mass transfer rate. Thus the assumption is that the right-hand side of Eq. (4) is always a function of f_w only, and the expression for θ is

$$\theta^2 = \frac{\alpha^2 \mu_e}{\rho_e u_e} \int_0^\infty C T^2 d\xi \quad (6)$$

in which $\alpha = 0.664$ and Γ^2 , evaluated for the similarity in conditions, is shown in Fig. 2. A good approximation to Γ^2 for helium injection is

$$\Gamma^2 = 1 + 1.500 f_w + 1.875 f_w^2 \quad (7)$$

Although this approach may be adequate if $(\rho v)_w$ is a reasonably smooth distribution [there are no data to check Eq. (6)], there is a difficulty if f_w vanishes on part of the surface, for then there are cases in which one finds an incorrect dependence for C_f on x . Examination of several problems^{27, 28} shows that c_{1w} should be used as the parameter C_f , i.e., Γ_0 (Fig. 3) is used in Eq. (6); the subscript on Γ_0 denotes

that it is to be regarded as a function of c_{1w} . On the other hand, $(\rho v)_w$ usually is given, and θ must be known to compute c_{1w} from formulas deduced later. One therefore apparently is forced to use f_w as a basis for Γ^2 in Eq. (6), at least as a first approximation; C_f/C_{f0} then can be found from Fig. 3.

This ambiguity perhaps is inevitable in an attempt to obtain relatively simple formulas for what is really quite a complicated flow problem. There appears to be no a priori justification, and in what follows θ is to be obtained from Eq. (6) by using the definition (5) of f_w and Fig. 2. Fortunately, as Fig. 2 shows, the momentum thickness is not particularly sensitive to surface mass transfer. The reason for this can be seen by noting that $\rho_e u_e^2 \theta$ is the difference in momentum flux between the flow in the boundary layer and a section of the external stream having the same mass flow. This difference is increased by $(\rho v)_w > 0$ and decreased by the associated reduction in wall shear stress; the two effects thus tend to balance. In addition, it will be seen later that the effect of any error in θ is diminished further because the influence of mass transfer on momentum thickness enters as a cube root in the $c_{1w}-(\rho v)_w$ and q_w-T_w relationships.

Relation between Surface Mass Transfer Rate and Concentration

When chemical reactions are unimportant and thermal diffusion is neglected, the equation governing the concentration of one of the species in a binary mixture boundary layer is

$$\rho \left[u \frac{\partial c_1}{\partial x} + v \frac{\partial c_1}{\partial y} \right] = \frac{\partial}{\partial y} \left(\frac{\mu}{Sc} \frac{\partial c_1}{\partial y} \right) \quad (8)$$

where Sc is the Schmidt number and c_1 will represent the concentration of the injected gas. Baron¹ has shown that thermal diffusion has negligible influence on the solution for c_1 ; its effect on the temperature distribution will be treated later. Associated with Eq. (8) is the boundary condition on mass transfer rate:¹

$$(\rho v)_w = - \frac{\mu_w}{Sc_w(1 - c_{1w})} \left(\frac{\partial c_1}{\partial y} \right)_w \quad (9)$$

Although it is attractive to regard the solution to Eq. (8) as simply a re-interpretation of the solution to the incompressible heat transfer problem, with q_w replaced by $(\rho v)_w$, $T_e - T_w$ by c_{1w} , etc., there are two conspicuous obstacles: q_w is related linearly to T_w , $q_w = -\kappa(\partial T/\partial y)_w$, but $(\rho v)_w$ is connected with c_{1w} by a nonlinear relation, and $(\rho v)_w$ appears explicitly in the integral momentum equation, whereas q_w does not. Consequently, any incompressible heat transfer results applied without change to the problem defined by Eqs. (8) and (9) can be valid at best for low injection rates. If the J - Δ method of Refs. 13-15 is applied to this problem, one should expect, even for $dp/dx = 0$, to obtain a family of curves, for various f_w , say, corresponding to the family obtained when $f_w = 0$ but $dp/dx \neq 0$. In this sense,

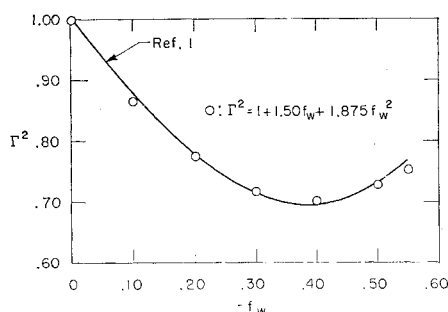


Fig. 2 Variation of Γ_0 with f_w

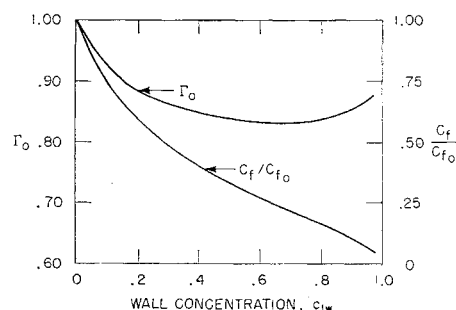


Fig. 3 Dependence of Γ_0 and C_f/C_{f0} on wall concentration of helium for similarity solutions¹

then, the problem of heat and mass transfer with zero pressure gradient is similar to the problem of heat transfer with variable pressure but no mass transfer through the surface.

The integral concentration equation, upon integrating Eq. (8) over the boundary layer, results in

$$(d/dx)(\rho_e u_e c_{1w}) = (\rho v)_w \quad (10)$$

$$\omega = \int_0^\infty \frac{\rho u}{\rho_e u_e} \frac{c_1}{c_{1w}} dy \quad (11)$$

Corresponding to the thermal thickness δ_T introduced by Hill, a concentration thickness δ_1 is defined as

$$\delta_1 = \int_0^\infty \frac{\rho}{\rho_w} \frac{c_1}{c_{1w}} dy \quad (12)$$

The ratio of densities ρ/ρ_w appears because the present treatment is for compressible flow. Then Eq. (10) can be rewritten for c_{1w} constant, and, by incorporating the boundary condition Eq. (9),

$$\frac{d}{dx} (\rho_e u_e \omega) = \frac{1}{c_{1w}} (\rho v)_w = \frac{C \mu_e}{C \lambda_w (\mu_e/\mu_w) \delta_1} \times \left[\frac{\lambda_w}{Sc_w(1 - c_{1w})} \frac{\delta_1}{c_{1w}} \left(- \frac{\partial c_1}{\partial y} \right)_w \right] \quad (13)$$

The quantity in brackets corresponds to w/Pr in Ref. 14. However, owing partly to the nonlinear boundary condition, the combination is here a function of c_{1w} and hence f_w , or $(\rho v)_w$, whereas w/Pr is independent of both $T_e - T_w$ and dp/dx . Even the part $(\delta_1/c_{1w})(-\partial c_1/\partial y)_w$ is not invariant because the concentration profiles are not the same for all f_w . Thus a departure from Hill's analysis is required. Define J_1 , Δ_1 analogous to J , Δ used before:¹⁴

$$J_1 = \frac{\omega}{\theta} \quad \Delta_1 = C \lambda_w \frac{\mu_e}{\mu_w} \frac{\delta_1}{\theta} \quad (14)$$

and set

$$n(c_{1w}) = \frac{\lambda_w}{Sc_w(1 - c_{1w})} \frac{\delta_1}{c_{1w}} \left(- \frac{\partial c_1}{\partial y} \right)_w \quad (15)$$

which is a known function of c_{1w} from the solution of Ref. 1. That is, the assumption is made that the solutions to the concentration equation belong to a one-parameter family identified by the wall concentration c_{1w} . The integral equation is now

$$(d/dx)(\rho_e u_e J_1) = C \mu_e n/\theta \Delta_1 \quad (16)$$

Equation (16) now can be integrated in the same manner as the energy equation in incompressible flow,¹³⁻¹⁵ since c_{1w} is constant. It is necessary only to compute a family of J_1 - Δ_1 curves in which each curve is identified by a value of c_{1w} rather than l . To do so, the foregoing integrations over y are changed to integrations over the modified Dorodnitsyn

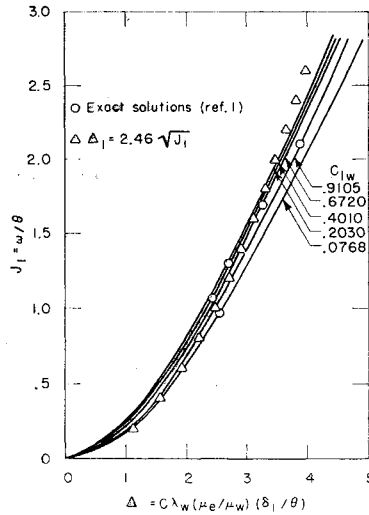


Fig. 4 Variation of J_1 with Δ_1 in a laminar boundary layer

variable η used by Baron:

$$d\eta = \left(\frac{1}{2x}\right) \left(\frac{Rx}{C}\right)^{1/2} \left(\frac{\rho}{\rho_e}\right) dy \quad (17)$$

Hence J_1 and Δ_1 become

$$J_1 = \frac{\int_0^\infty \frac{u}{u_e} \frac{c_1}{c_{1w}} d\eta}{\int_0^\infty \frac{u}{u_e} \left(1 - \frac{u}{u_e}\right) d\eta} = J_1(\Delta_1, c_{1w})$$

$$\Delta_1 = \frac{\int_0^\infty \frac{c_1}{c_{1w}} d\eta}{\int_0^\infty \frac{u}{u_e} \left(1 - \frac{u}{u_e}\right) d\eta}$$

All of the foregoing integrands are functions of η only for the similarity solutions. It is clear that in the present problem $J_1(\Delta_1, c_{1w})$ corresponds to $J(\Delta, l)$ in the constant property heat transfer analysis. By varying Δ_1 , curves of J_1 vs Δ_1 can be computed for each value of c_{1w} . Although construction of this family of curves requires numerical integration, the special points (J_{1s}, Δ_{1s}) corresponding to the similarity solutions tabulated by Baron can be found without difficulty by direct integration of the momentum and concentration equations for θ and ω . For c_{1w} fixed, the c_1/c_{1w} profiles then are scaled in the ratio Δ_1/Δ_{1s} to cover the entire range of Δ_1 , while maintaining their shape and the velocity profiles invariant. Figure 4 shows these results.

Stepwise integration of Eq. (16) can be avoided by using the approximation to the entire family:

$$\Delta_1 = a_1 J_1^{1/2} \quad (18)$$

which is shown in Fig. 4 with $a_1 = 2.46$ [see Eq. (24)]. Thus Eq. (16) yields

$$J_1 = \left(\frac{3n}{2a_1}\right)^{2/3} \left(\frac{\mu_e}{\rho_e u_e}\right)^{2/3} \frac{1}{\theta} \left[\int_0^x \frac{C}{\theta^{1/2}} dx\right]^{2/3}$$

for $J_1 = 0$ at $x = 0$. When c_{1w} is constant, $(\rho v)_w$ therefore is given by

$$\frac{(\rho v)_w}{\rho_e u_e} = \frac{C \mu_e n c_{1w}}{\theta \Delta_1} = \left(\frac{2}{3a_1^2}\right)^{1/3} \left(\frac{\mu_e}{\rho_e u_e}\right)^{2/3} \frac{C}{\theta^{1/2}} \times \left[\int_0^x \frac{C}{\theta^{1/2}} dx\right]^{-1/3} (c_{1w} n)^{2/3} \quad (19)$$

This relation between c_{1w} and $(\rho v)_w$ is valid only for c_{1w} constant and therefore is limited severely. Superposition is strictly prevented by the dependence of θ on c_{1w} . However, because θ is affected only moderately by mass transfer,

elementary solutions of the form (19) may be superposed approximately to give a formula for $(\rho v)_w$ when c_{1w} is variable. The approximation is improved by splitting off the part of θ depending on c_{1w} when c_{1w} is constant, that is, by dividing θ by Γ_0 :

$$\frac{(\rho v)_w}{\rho_e u_e} = \left(\frac{2}{3a_1^2}\right)^{1/3} \left(\frac{\mu_e}{\rho_e u_e}\right)^{2/3} \frac{C}{(\theta/\Gamma_0)^{1/2}} \times \left[\int_0^x \frac{C}{(\theta/\Gamma_0)^{1/2}} dx\right]^{-1/3} g \quad (20)$$

Thus, to the extent that one may ignore the influence of c_{1w} on θ/Γ_0 , $(\rho v)_w$ is related linearly to $g = c_{1w} n^{2/3} \Gamma_0^{1/3}$. One also can show easily, by using the foregoing formula for J_1 , that Eq. (10) is a linear relation between $(\rho v)_w$ and g , provided that variations of θ/Γ_0 with c_{1w} are neglected. Since θ/Γ_0 will not vary greatly when c_{1w} is not constant, changes in θ due to mass transfer will have indeed a weak influence on the relationship between $(\rho v)_w$ and c_{1w} . Hence, approximations in the solution for θ from the momentum equation when $dp/dx = 0$ have only a minor effect on the results obtained from the concentration equation.

A particular distribution $(\rho v)_w$ therefore may be constructed by superposing increments in $(\rho v)_w$ due to increments in g , and for g variable in the range $0 \leq \xi \leq x$:

$$\frac{(\rho v)_w}{\rho_e u_e} = \left(\frac{2}{3a_1^2}\right)^{1/3} \left(\frac{\mu_e}{\rho_e u_e}\right)^{2/3} \frac{C}{(\theta/\Gamma_0)^{1/2}} \times \left\{ (g)_{x=0} + \left[\int_0^x \frac{C}{(\theta/\Gamma_0)^{1/2}} dx_1\right]^{-1/3} + \int_0^x \left[\int_{\xi}^x \frac{C}{(\theta/\Gamma_0)^{1/2}} dx\right]^{-1/3} \frac{dg}{d\xi} d\xi \right\} \quad (21)$$

Hence with g constant, and θ/Γ_0 for the similarity conditions given by

$$\theta/\Gamma_0 = \alpha (C \mu_e / \rho_e u_e)^{1/2} x^{1/2} \quad (22)$$

Eq. (22) eventually leads to

$$-f_w = (4/\alpha a_1^2)^{1/3} g \quad (23)$$

It happens that quite good agreement with the numerical solutions is obtained if a_1 satisfies

$$a_1 = (4/\alpha)^{1/2} = 2.46 \quad (24)$$

Thus the function $g(c_{1w})$ reduces exactly to $-f_w$ when c_{1w} is constant; comparison between g and the f_w - c_{1w} relation computed in Ref. 1 is shown in Fig. 5. This constitutes the first check of the various assumptions introduced previously and is favorable except for $-f_w$ greater than 0.5. It is to be noted that the function $g(c_{1w})$, the solid line in Fig. 5 and

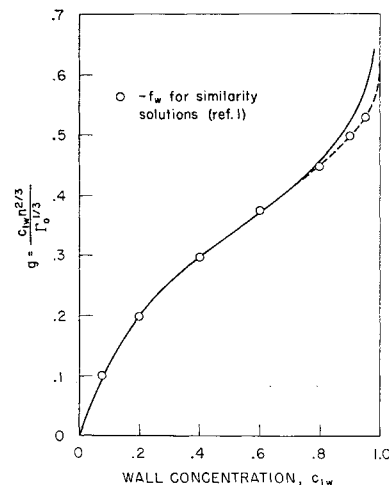


Fig. 5 Variation of the function g with wall concentration of helium: comparison of exact and approximate solutions

likewise computed from Ref. 1, is assumed to hold generally; it reduces to $-f_w$ only when c_{1w} and f_w do not depend on x .

Equation (20) can be solved for g (see Appendix), and the solution to the integral concentration equation is represented by the pair of integrals

$$\frac{(\rho v)_w}{\rho_e u_e} = \frac{\kappa_1 c}{(\theta/\Gamma_0)^{1/2}} \int_{x_0}^x \left[\int_{\xi}^x \frac{C}{(\theta/\Gamma_0)^{1/2}} dx_1 \right]^{-1/3} \frac{dg}{d\xi} d\xi \quad (25)$$

$$g(x) - g(x_0^-) = \frac{\sin \pi/3}{\kappa_1 \pi} \int_{x_0^+}^x \left[\int_{\xi}^x \frac{C}{(\theta/\Gamma_0)^{1/2}} dx_1 \right]^{-2/3} \times \frac{(\rho v)_w}{\rho_e u_e} d\xi \quad (26)$$

$$\kappa_1 = (2/3 a_1^2)^{1/3} (\mu_e / \rho_e u_e)^{2/3} \quad (27)$$

Relation between Surface Mass Transfer, Heat Transfer, and Temperature

Mass and heat transfer at the surface can be related to temperature in a form similar to Eqs. (25) and (26) by applying the same method of solution to the integral energy equation. Although thermal diffusion has a significant influence on the solution to the energy equation, the effect enters in such a way that, as later remarks will clarify, it can be ignored for the present; the corresponding integral energy equation is

$$\frac{d}{dx} \left\{ \rho_e u_e \int_0^\infty \frac{\rho u}{\rho_e u_e} (h_{0e} - h_0) dy \right\} = \frac{\mu_w}{Pr_w} \left(\frac{\partial h_0}{\partial y} \right)_w + (\rho v)_w (h_{0e} - h_w) + (\rho v)_w (1 - c_{1w}) (C_{p12} - 1) \frac{T_w}{T_{0e}} \left(\frac{Sc_w}{Pr_w} - 1 \right) \quad (28)$$

Although sometimes used, this form of the equation is not the most convenient for the present work. Owing to the dependence of C_p on composition, the differential of enthalpy is given by

$$dh = C_p dT + C_{p2} (C_{p12} - 1) T dc_1 \quad (29)$$

with $C_{p12} = C_{p1}/C_{p2}$. After using Eq. (29) and the integral concentration equation, one can write Eq. (28) in the form

$$\frac{d}{dx} \left\{ \rho_e u_e \int_0^\infty \frac{\rho u}{\rho_e u_e} C_p (T_{0e} - T_0) dy \right\} = \left(\kappa \frac{\partial T}{\partial y} \right)_w + (\rho v)_w C_{p1} (T_{0e} - T_w) \quad (30)$$

Thus, the heat flux to the surface, $(\kappa \partial T / \partial y)_w$, is separated explicitly from the energy transport by diffusion and convection. An additional advantage of Eq. (30) is that the integral contains the temperature difference rather than the enthalpy difference from the freestream value. Whereas stagnation values arise rather naturally in the integral energy equation (28), the numerical solutions are obtained more easily for static temperatures, so that further argument is necessary if one is to use tabulated numerical results directly as an aid to solution of Eq. (30). It can be shown^{1, 28} that, if variations in λ , C_p , the Prandtl number Pr , and the Schmidt number Sc with temperature are much smaller than variations with concentration, then one can superpose solutions (for temperature) to the differential energy equation. This implies, in particular, that as in homogeneous flows, the "temperature potential" in high speed flow must be measured from the adiabatic wall temperature, T_{aw} . It follows further²⁸ that one can obtain the solution for heat transfer in high speed flow from the result for low speed flow simply by substituting the proper temperature potential.

It is sufficient, then, to solve the integral equation (30) in (x, η) coordinates for low speed flow, T_e and T replacing T_{0e} and T_0 in Eq. (30):

$$\frac{d}{dx} \left\{ \rho_e u_e \int_0^\infty \frac{\rho u}{\rho_e u_e} C_p (T_e - T) dy \right\} = \left(\kappa \frac{\partial T}{\partial y} \right)_w + (\rho v)_w C_{p1} (T_e - T_w) \quad (31)$$

Then, in the final results for heat transfer, the temperature potential $T_w - T_e$ is to be replaced by $T_w - T_{aw}$, where T_{aw} is known from the similarity solutions or from experiments; no attempt will be made to determine the particular solution T_{aw} . The appropriate values to use for T_{aw} are best determined separately for each problem considered and are set by the obvious requirement that $T_w = T_{aw}$ when $(\partial T / \partial y)_w$ vanishes under the conditions supposed. If, on the other hand, thermal diffusion is not neglected, $(\partial T / \partial y)_w$ is coupled to T_w through the mass transfer rate^{1, 31} even when $T_w = T_{aw}$. This appears to be the most important influence of thermal diffusion; its effect on the normalized heat transfer, in the form of the Stanton number, seems to be very small.³¹ Hence, neglect of thermal diffusion to obtain the integral equations may not be a serious defect, providing the proper adiabatic wall temperature can be incorporated in the final results. This is not a trivial matter, since T_{aw} is rather strongly dependent on mass transfer rate as well as Mach number. The corresponding complication is absent from heat transfer problems not involving mass transfer but with variable pressure, for then T_{aw} can be assumed with good accuracy to be independent of variations in pressure.

Now consider Eq. (31) when $(\partial T / \partial y)_w \neq 0$ and for an isothermal wall along which c_{1w} is constant; it then is permissible to divide the equation by $C_{pw}(T_e - T_w)$ to obtain

$$\frac{d}{dx} (\rho_e u_e \delta_c) = \frac{\mu_w}{Pr_w} \frac{1}{T_e - T_w} \left(\frac{\partial T}{\partial y} \right)_w + \frac{C_{p1}}{C_{pw}} (\rho v)_w \quad (32)$$

where now

$$\delta_c = \int_0^\infty \frac{\rho u}{\rho_e u_e} \frac{C_p}{C_{pw}} \left(\frac{T - T_e}{T_w - T_e} \right) dy \quad (33)$$

Define a thermal thickness by

$$\delta_T = \int_0^\infty \frac{\rho C_p}{\rho_e C_{pw}} \left(\frac{T - T_e}{T_w - T_e} \right) dy \quad (34)$$

in which the ratio $\rho C_p / \rho_e C_{pw}$ appears because both ρ and C_p vary through the boundary layer. After inserting Eq. (9) for $(\rho v)_w$, the equation can be put in a normalized form as

$$(d/dx)(\rho_e u_e \theta J) = C_{\mu} \lambda_w W / \theta \Delta \quad (35)$$

with the definitions analogous to quantities introduced before:

$$J = \frac{\delta_c}{\theta} \quad \Delta = C_{\lambda_w} \frac{\mu_e}{\mu_w} \frac{\delta_T}{\theta} \quad (36)$$

$$W = \frac{1}{Pr_w} \frac{\delta_T}{T_e - T_w} \left(\frac{\partial T}{\partial y} \right)_w + \frac{c_{1w}}{1 - c_{1w}} \times \frac{C_{p1}/C_{pw}}{Sc_w} \frac{\delta_T}{c_{1w}} \left(- \frac{\partial c_1}{\partial y} \right)_w \quad (37)$$

The group of terms $\lambda_w W$ corresponds to n in Eq. (16) and is a function of c_{1w} only; for the special conditions c_{1w} and T_w both constant, it can be evaluated from the numerical results of Ref. 1. Figure 6 shows the more useful quantity

$$\phi = \frac{C_{pw}}{C_{p2}} \frac{(\lambda_w W)^{2/3}}{(2a_2^2 \alpha \Gamma_0)^{1/3}} \quad (38)$$

where $a_2 = 2.59$ in the approximation $J = a_2 \Delta^{1/2}$. It fol-

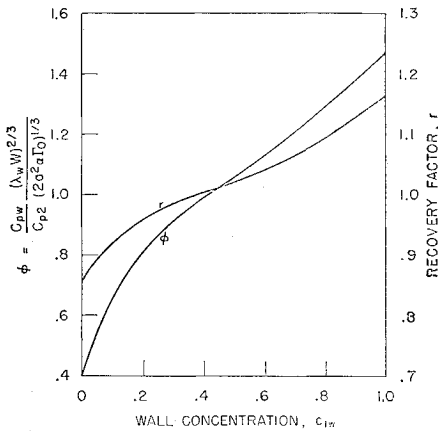


Fig. 6 Variation of ϕ and recovery factor r with wall concentration of helium for similarity solutions

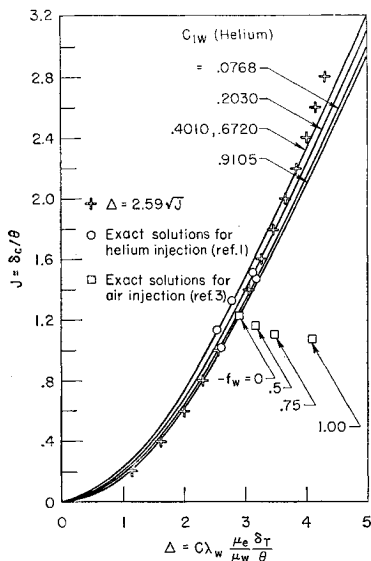


Fig. 7 Variation of J with Δ in a laminar boundary layer with $dp/dx = 0$, $(\rho v)_w \pm 0$

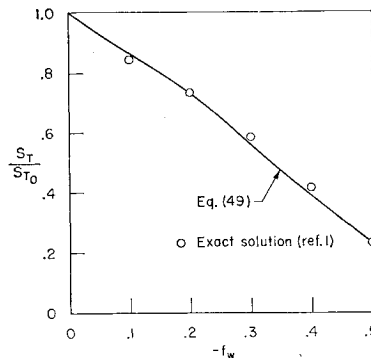


Fig. 8 Variation of Stanton number with mass transfer rate: comparison of exact and approximate solutions

lows from previous remarks²⁸ that the numerical results of Ref. 1 may be used to compute a family of J - Δ curves for the integral energy equation; Fig. 7 shows these results for helium injection. Eventually, one can deduce the formula

$$Q_w = \left(\frac{2}{3a_2^2} \right)^{1/3} \frac{[\mu_e(\rho_e u_e)^{1/2}]^{2/3} C}{(\theta/\Gamma_0)^{1/2}} \left[\int_0^x \frac{C}{(\theta/\Gamma_0)^{1/2}} dx \right] \Sigma \quad (39)$$

if the family in Fig. 7 is approximated by $J = a_2 \Delta^{1/2}$, and with

$$\Sigma = (T_{aw} - T_w) C_{pw} (\lambda_w W)^{2/3} / \Gamma_0^{1/3}$$

Q_w is an abbreviation for (see Ref. 28)

$$Q_w = q_w + \frac{1}{2} C_{p1} g \rho_e u_e (C/R_x)^{1/2} (T_{aw} - T_w) \quad (40)$$

where $q_w = [x(\partial T/\partial y)]$. In accord with previous discus-

sion, the correct temperature potential, $T_{aw} - T_w$, has been inserted for $T_e - T_w$ so that Eq. (39) holds for $Me \neq 0$. When $(\rho v)_w = 0$ locally, one must²⁸ take $Q_w = q_w$.

Equation (39) is a linear relation between Q_w and Σ ; when variations in θ/Γ_0 with c_{1w} are ignored, the multiplier of Σ is a function of x only. One can show also that the differential equation (35) is a linear relation between Q_w and Σ , so that, by superposition, an expression for Q_w valid for variable surface concentration and temperature may be constructed:

$$Q_w = \left(\frac{2}{3a_2^2} \right)^{1/3} \frac{[\mu_e(\rho_e u_e)^{1/2}]^{2/3} C}{(\theta/\Gamma_0)^{1/2}} \times \int_0^x \left[\int_\xi^x \frac{C}{(\theta/\Gamma_0)^{1/2}} dx \right]^{-1/3} \frac{d\Sigma}{d\xi} d\xi \quad (41)$$

It should be noted that T_{aw} in Σ is a local value and varies with wall concentration in a manner to be discussed later. To be consistent with the viewpoint taken here, it is assumed that, just as for $g(c_{1w})$, the dependence of Σ on c_{1w} as determined from special numerical results remains the same irrespective of the distributions of c_{1w} and $T_{aw} - T_w$ along the surface. Once again, only subsequent comparison with numerical results and experimental data can justify this assumption.

The first check of this formula is comparison with the heat transfer computed by Baron. From Eqs. (40) and (41) with Σ constant, the normalized heat transfer (Stanton number) is

$$St = \frac{g_w}{\rho_e u_e C_{p2} (T_{aw} - T_w)} = \left(\frac{2}{3a_2^2} \right)^{2/3} \left(\frac{C u_e}{\rho_e u_e} \right)^{2/3} \frac{\phi}{(\theta/\Gamma_0)^{1/2}} \left[\int_0^x \frac{dx_1}{(\theta/\Gamma_0)^{1/2}} \right]^{-1/3} - \frac{1}{2} \frac{C_{p1}}{C_{p2}} g \left(\frac{C}{R_x} \right)^{1/2} \quad (42)$$

Since θ/Γ_0 is given by the expression (22), the Stanton number is finally

$$St(R_x/C)^{1/2} = \phi - \frac{1}{2} (C_{p1}/C_{p2}) g \quad (43)$$

This reduces to the correct value for no mass transfer if

$$a_2 = 2w/\alpha^2 = 2.59 \quad (44)$$

where $w = 0.571$ is the value of W when $c_{1w} = f_w = 0$ everywhere and has been encountered previously in Hill's analysis. Note that this value of a_2 also permits $J = a_2 \Delta^{1/2}$ to be a good approximation to the J - Δ family in Fig. 7. Equation (43) with ϕ taken from Fig. 6 is shown in Fig. 8 with the exact values of Ref. 1; in this particular case, of course, g and $-f_w$ are interchangeable.

Evidently, then, replacement of T_e by T_{aw} in the temperature potential does account for the changes in adiabatic wall temperature due to both Mach number and mass transfer, as earlier argument showed. The present method will yield no information concerning the temperature of a surface that is insulated over its entire length. The importance of the results obtained here resides in the possibility of carrying out calculations of heat transfer and wall temperatures under conditions that are not treated easily by numerical solution to the differential equations.

Several interesting features of Eq. (43) should be noted. The term denoted by ϕ is related to the amount of energy "convected" within the boundary layer; this is increased by mass transfer (cf., Fig. 6) because energy losses to the surface thereby are reduced and because there is some enthalpy addition associated with the mass transferred through the surface. However, so far as surface heat transfer is concerned, the increased energy content of the boundary layer is more than balanced by convection and diffusion away

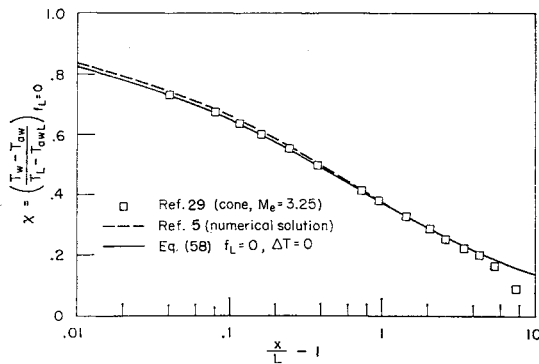


Fig. 9 Comparison of approximate result with numerical solution and experimental data for upstream cooling with no surface mass transfer

from the surface represented by the second term in Eq. (43). Not only is the latter influence proportional to the mass transfer rate (recall that, locally, $g = -f_w \sim (\rho v)_w$ for the similarity conditions), but the importance of using an injected gas having a high specific heat also is indicated clearly. In consequence of the two opposing effects shown in Eq. (43), the temperature gradient at the surface, and hence q_w , is reduced by mass transfer.

Equation (41) can be solved for Σ according to the Appendix, and one finally has the two relations corresponding to Eqs. (25) and (26):

$$Q_w = \frac{\kappa_2 C}{(\theta/\Gamma_0)^{1/2}} \int_{x_0}^x \left[\int_{\xi}^x \frac{C dx_1}{(\theta/\Gamma_0)^{1/2}} \right]^{-1/3} \frac{d\Sigma}{d\xi} d\xi \quad (45)$$

$$\Sigma - \Sigma(x_0^-) = \frac{\sin \pi/3}{\kappa_2 \pi} \int_{x_0}^x \left[\int_{\xi}^x \frac{C dx_1}{(\theta/\Gamma_0)^{1/2}} \right]^{-2/3} Q_w d\xi \quad (46)$$

$$\kappa_2 = \left(\frac{2}{3a_2^2} \right)^{1/3} [\mu_e (\rho_e u_e)^{1/2}]^{2/3} \quad (47)$$

Q_w is given by Eq. (40) and $\Sigma = C_{pw}(T_{aw} - T_w)(\lambda_w W)^{2/3}/\Gamma_0^{1/3}$. These formulas contain, as a special case, Hill's result¹⁴ for heat transfer to a flat plate with nonuniform surface temperature and no mass transfer.

In summary, the formulas deduced here for problems involving simultaneous heat and mass transfer are to be used in the following manner:

1) Suppose $(\rho v)_w$ is given as a function of x ; then Eq. (5) is used to compute $f_w(x)$, and by using either Fig. 2 or Eq. (7), the momentum thickness θ is calculated from Eq. (6). Here, and in some later steps, it may be necessary to use an estimated average value for C , found from Fig. 1.

2) The function $g(x)$ is calculated from Eq. (26), and the distribution of wall concentration then is read off Fig. 6.

3) If the surface temperature is specified, Eqs. (45) and (40) give the surface heat transfer q_w , and if q_w is specified, the surface temperature can be calculated from Eq. (46). The results of step 2 and Fig. 6 are required here; the dependence of T_{aw} on c_{1w} is treated in the next section. In particular, note again that $Q_w = q_w$ wherever $(\rho v)_w$ vanishes; otherwise Eq. (40) holds.

Application to Discontinuous Distributions of Surface Mass Transfer

The first problem concerns the influence of a solid tip. For structural reasons, there always must be a finite solid region at the leading edge. Consider the distribution with $f_{w\infty}$ constant:

$$f_w = \begin{cases} 0 & 0 \leq x < L \\ f_{w\infty} & x > L \end{cases}$$

Note that $(\rho v)_w$ is finite at $x = L$; the "singularity" is still

at $x = 0$ [$(\rho v)_w \sim x^{-1/2}$], which lies outside the mass transfer region. Even though f_w is constant downstream of $x = L$, c_{1w} is not; the problem is to compute the growth of c_{1w} up to its value $c_{1\infty}$ associated with $f_{w\infty}$ under similarity conditions. It follows from Eq. (6) that the momentum thickness is given by

$$\theta = \alpha \left(\frac{C \mu_e}{\rho_e u_e} \right)^{1/2} \begin{cases} x^{1/2} & 0 \leq x < L \\ \Gamma_0(l_1 + x)^{1/2} & x > L \end{cases}$$

With some approximations that are valid for $c_{1w} \rightarrow 0$, in particular $l_1 \ll x, \xi$, one finds²⁷ from Eq. (26)

$$\frac{g(c_{1w})}{-f_{w\infty}} = \frac{3}{4} \frac{\sin \pi/3}{\pi} \int_L^x \frac{d\xi}{\xi^{1/2} (x^{3/4} - \xi^{3/4})^{2/3}} \quad (48)$$

and with the transformation $\zeta = (\xi/x)^{3/4}$

$$g(c_{1w})/-f_{w\infty} = 1 - \chi(x/L) \quad (49)$$

where

$$\chi \left(\frac{x}{L} \right) = \frac{\sin \pi/3}{\pi} \int_0^{(L/x)^{3/4}} \frac{d\zeta}{\zeta^{1/3} (1 - \zeta)^{2/3}} \quad (50)$$

The function χ , an incomplete beta function, has been encountered previously in a calculation of the distribution of adiabatic wall temperature downstream of a cooled leading edge.²⁹ It can be evaluated by expanding the integrand about $\zeta = 0$ and $\zeta = 1$ and also has been tabulated, apart from a constant multiplier, in Ref. 30. Figure 9 shows χ , and in Fig. 10 the result Eq. (49) is compared, by using Fig. 5, with the numerical result of Ref. 4. Agreement seems to be acceptable in the range for which numerical calculations have been carried out. Owing to the approximations made here, the expression $1 - \chi$ quite likely lies below the true result for larger values of x/L .

A much more useful solution can be obtained for the upstream cooling problem, for which there appears to be no numerical solution available in the case of a binary mixture. (The numerical procedure of Ref. 4 can be used to solve this problem.) Suppose that the distribution of f_w is

$$f_w = \begin{cases} f_{wL} & 0 \leq x < L \\ 0 & x > L \end{cases}$$

Equation (26) eventually yields

$$\frac{g}{-f_{wL}} = \frac{\sin \pi/3}{\pi} \int_0^{(L/x)^{3/4}} \frac{d\zeta}{\zeta^{1/3} (1 - \zeta)^{2/3}} = \chi \left(\frac{x}{L} \right) \quad (51)$$

Thus, by using Fig. 5, the distribution of c_{1w} for $x/L > 1$ can be found easily.

The simplest idealized form of the upstream cooling problem consists in determining the temperature of an insulated ($q_w = 0$) impermeable surface downstream of a porous region in which the similarity conditions are met. However,

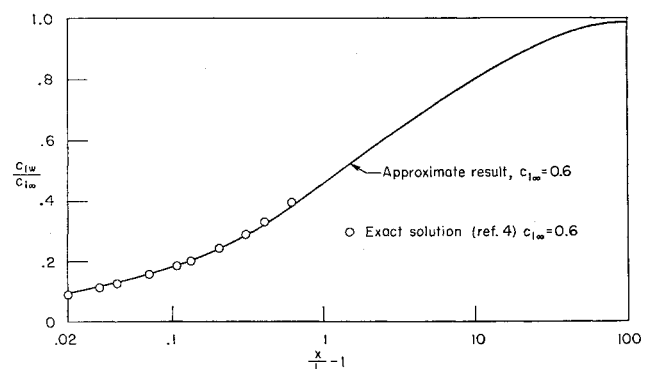


Fig. 10 Variation of $c_{1w}/c_{1\infty}$ over injection region downstream of a solid leading edge (flat plate)

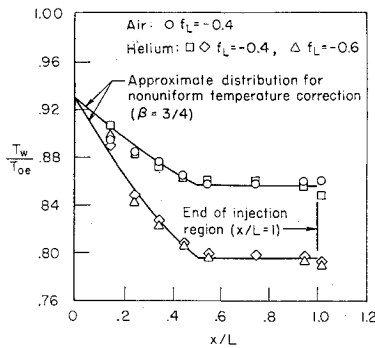


Fig. 11 Experimental surface temperature in porous region of cone³³

when measurements are taken, two violations of these requirements are difficult to avoid: there is a solid leading edge so that $f_w = 0$ for some distance near $x = 0$, and, because cooling therefore is not present in this region, the temperature tends to be nonuniform. The first departure from the ideal situation will be neglected for the present, but a correction for nonuniform temperature is included. A simple approximation to the actual conditions for $0 < x < L$ is then (see Fig. 11)

$$f_w = \begin{cases} f_{wL} & 0 \leq x < L \\ 0 & x > L \end{cases}$$

$$T_w = T_L + \begin{cases} \Delta T \{1 - (x/L)^\beta\} & 0 < x < L \\ 0 & L < x < L \end{cases}$$

Equations (45) and (46) lead to

$$\psi = \frac{T_w - T_{aw}}{T_L - T_{awL}} = \frac{\phi_L}{\phi} \left\{ \chi + \frac{\Delta T}{T_L - T_{awL}} (\chi - \tau) \right\} \quad (52)$$

where τ is a function that can be evaluated²⁸ in much the same way as χ , and T_{awL} is the adiabatic wall temperature in the porous region.

Within the approximations used here, ψ is independent of temperature level and varies with f_{wL} (and, of course, x/L). Unfortunately, this is not a convenient quantity to compare with experimental results, since the adiabatic wall temperatures, T_{awL} and T_{aw} , have been quite difficult to measure accurately. A detailed discussion and comparison of theoretical and experimental values of adiabatic wall temperatures is given in Ref. 31; it is shown that there is perhaps some reason to question the accuracy (or interpretation) of measured values. Of course it is possible in principle to measure both T_{awL} and T_{aw} in Eq. (52), since, when $q_{wL} = 0$ for all x , then $T_L = T_{awL}$ and $T_w = T_{aw}$, the resulting values then could be used in Eq. (52) when $q_w \neq 0$. In view of the doubts just mentioned, however, it seems best at present to compute T_w from Eq. (52), and hence only measurements of surface temperature when $q_{wL} \neq 0$ will be required for comparison.

It is still necessary to know T_{awL} and T_{aw} , which, according to earlier remarks, cannot be computed by the present method. Hence one must treat the adiabatic wall temperature, or equivalently the recovery factor r , in the same way as ϕ was handled before. For a consistent treatment, the values of r computed in Ref. 1, neglecting thermal diffusion, should be used here, since the analysis depends in part on a linear energy equation. There is, however, a significant error between the results of Ref. 1 and measurements. More recent calculations³¹ accounting for thermal diffusion show that the error between theory and experiment is still significant but of opposite sign. Both theoretical results and four sets of experimental points^{32, 35-37} are shown in Fig. 12. The calculations including thermal diffusion are to be preferred, since the error appears to be associated with interpretation of the conditions actually existing in the experiments; the difference between the data and the results when

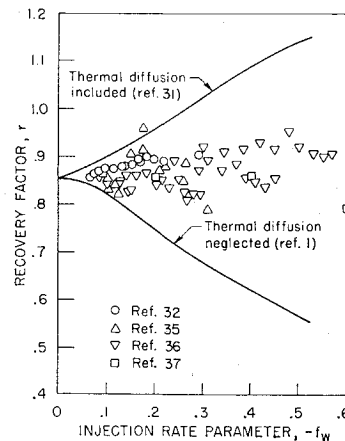


Fig. 12 Variation of recovery factor with mass transfer rate³¹

thermal diffusion is neglected apparently cannot be explained so easily.³¹

Accordingly, the relationship between r and c_{1w} used here is based on the results of Ref. 31 and is given in Fig. 6. Thermal diffusion has little influence on the variations of St and c_{1w} with f_w , so that the approximate results obtained from the energy equation will be modified only in the temperature potential $T_w - T_{aw}$. It is assumed, as for ϕ , that the dependence shown in Fig. 6 holds for any T_w and c_{1w} distributions.

All quantities in Eqs. (52) and (51) now are known; the distributions of r and ϕ with x/L are found from Figs. 5 and 6. A comparison with the measurements of Ref. 33 for three experimental conditions is shown in Fig. 13. The temperature T_L is that measured at a thermocouple immediately upstream of the end of the injection region. It happens for the cases computed that $\chi - \tau < 0.03$ always. The dashed line for the case $f_{wL} = -0.6$, $T_L/T_{oe} = 0.791$ includes both a correction for nonuniform temperature and adjustment to account for the fact noted previously, that $g(c_{1w})$ in Fig. 5 deviates from the correct values computed in Ref. 1 for large c_{1w} . Thus the dashed line in Fig. 13 for $f_L = -0.6$ is based on the dashed line in Fig. 5. This correction, which affects principally c_{1L} , ϕ_L , and T_L , is negative and of larger magnitude than the correction for nonuniform surface temperature, so that the corrected curve lies below the uncorrected result.

The calculated surface temperatures seem to agree satisfactorily with the measurements for the three cases shown.† A discussion of the expected experimental accuracy may be found in Ref. 33. Note that greater error in the computed

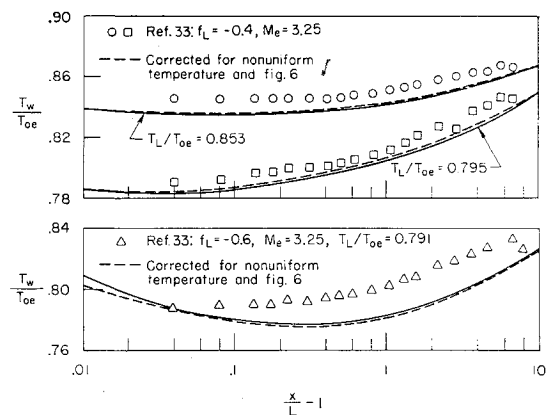


Fig. 13 Comparison of approximate results and experimental data for upstream cooling problem: helium injection

† Since the measurements were made on a cone, the Mangler transformation has been used to reduce them to the corresponding flat plate values shown in Fig. 12.

results must be expected for $f_L = -0.6$, since the solutions¹ on which this analysis is based do not extend to values of f_w this large; as the mass transfer rate becomes large, $(\partial u / \partial y)_w \rightarrow 0$, and separation is predicted by the numerical results.

A correction for the solid tip on the cone should be included, but to do so accurately requires considerably more labor than seems worthwhile at present. The reason is that the calculation carried out previously for Fig. 10 must be done accurately for a distance in the porous region nine times the length of the solid tip (i.e., the porous region on the cone used in the experiments was nine times the length of the solid tip). This will lead to rather lengthy calculations, for the sole purpose of obtaining the value c_{1L} of c_{1w} at the downstream end of the porous region; the values of c_{1w} for $x/L > 1$ can be obtained easily. One can estimate c_{1L} and thereby determine that the consequent correction to the curves in Fig. 13 certainly will not exceed the evident difference between experiment and the approximate results. Hence, tentatively, one might regard the disagreement between experiment and (approximate) theory shown here as a reflection of the effect of the solid tip. Note that the influence seems to be relatively greater for the larger f_{wL} , as one might expect.

The assumption that ϕ and r are the same functions of c_{1w} for $x > L$, where $f_w = 0$, as for f_w constant therefore apparently is supported. Additionally, the present results contribute some evidence favoring the argument in Ref. 31 that thermal diffusion may have substantial influence on recovery temperatures when helium is injected into air. In the problem considered here, the $r - c_{1w}$ relationship of Fig. 6 (i.e., thermal diffusion included) has the trend necessary for theoretical prediction of an apparently peculiar behavior downstream of the injection region: the surface temperature actually decreases before increasing eventually toward the adiabatic wall temperature associated with $c_{1w} = 0$ at $x \rightarrow \infty$.[†] This initial reduction in T_w evidently is associated with the decrease in T_{aw} as c_{1w} decreases downstream of $x = l$, for T_{aw} is the surface temperature when $q_w = 0$ for all x , and the difference between T_w and T_{aw} for $x > L$ is roughly proportional to q_w (i.e., the energy added to the boundary layer) in the cooled region. Consequently, the behavior of T_{aw} for $x > L$ should be reflected in T_w , as the measurements have shown.

Some Results for Air Injection

The results presented so far can be specialized to the case of air injection, but there are several interesting difficulties. A formula similar to Eq. (22) holds for the momentum thickness; since $\lambda_w = 1$ for air, the Stanton number for similarity conditions becomes, as a special case of Eq. (43),

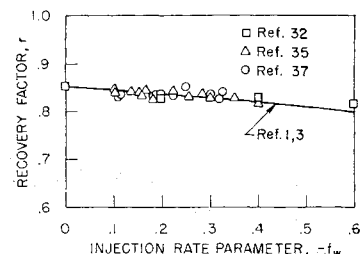
$$St \left(\frac{Rx}{c} \right)^{1/2} = \frac{W_a^{2/3}}{(2a_2^2 \alpha \Gamma_{a0})^{1/3}} + \frac{1}{2} f_w = \phi_a + \frac{1}{2} f_w \quad (53)$$

$$\Gamma_{a0} = \frac{Cf}{Cf_0} - \frac{1}{\alpha} f_w$$

where subscript a denotes air injection. The numerical solutions of Refs. 1 and 3 can be used for air injection in the same way that the results of Ref. 1 formed the basis for helium injection. For $a_2 = 2.59$, Eq. (53) is more inaccurate the larger is f_w . The reason for this can be seen in Fig. 7. It is clear that the assumption $\Delta = 2.59 J^{1/2}$, $J = a_2 \Delta^{1/2}$, is not

[†] The behavior seen here is for $T_w < (T_{aw})_\infty$; if $(T_{aw})_\infty < T_w$, then the surface temperature would decrease monotonically from T_w to $(T_{aw})_\infty$, although there are no measurements taken under the proper conditions to show this. For air injection, when thermal diffusion is absent, the initial decrease is not observed (see Fig. 16).

Fig. 14 Variations of recovery factor with mass transfer rate for air injection⁸



a good approximation to the J - Δ family for air injection, and the error obviously is greater for larger values of f_w . This does not mean that the method of solution will not work; it does mean that the approximation is representative only over narrow ranges of f_w . The value of a_2 can be adjusted to match the range of f_w in a particular problem, but if the range is large, then $J = a_2 \Delta^{1/2}$ is a poor choice, and Eq. (35) should be integrated in steps, using the J - Δ family of curves. Thus the simplest form of the J - Δ relationship is in a sense fortuitous and depends on the circumstance that the similarity solutions lie close to the line $\Delta = a_2 J^{1/2}$ or, more generally, $\Delta = a_2 J^\epsilon$, where $\epsilon \geq 0$.

It is interesting, however, that Eq. (52) is independent of a_2 ; hence the difficulty just mentioned apparently is irrelevant, and one might expect to obtain reasonable results for air injection in the upstream cooling problem. Equation (52) then becomes, if one ignores the correction for nonuniform surface temperature,

$$\frac{T_w - T_{aw}}{T_L - T_{awL}} = \frac{\phi_{aL}}{\phi_a} \chi \left(\frac{x}{L} \right) \quad (54)$$

from which the surface temperature may be calculated.

Since there is only a single gas, thermal diffusion is not an issue; correspondingly, the theoretical prediction for the recovery factor agrees very well with measurements, as Fig. 14 taken from Ref. 31 shows, and T_{aw} certainly can be calculated accurately. However, a difficulty of a different sort arises because now the term "wall concentration" has doubtful meaning. In the calculations for helium injection, c_{1w} was very important, since it served essentially as a parameter identifying (by assumption) the values of r and ϕ at a particular position $x/L > 1$. Without c_{1w} , there is no obvious way of connecting the values of r and ϕ_a computed from the similarity solutions, with the values required in Eq. (54).

Therefore, in the absence of an alternative, it will be supposed here that, even with air injection into an air boundary layer, one can identify a quantity called the concentration of injected gas. This amounts to attaching some special significance to the particles that have been injected through the surface. Although evidently it is not consistent with ideas regarding the indistinguishability of identical molecules, this is a formal artifice that seems to be quite effective for present purposes.²⁸ The results are shown in Fig. 15.

Figure 16 shows a comparison of the approximate result for T_w with experimental data reported in Ref. 33 and with some numerical results of Ref. 5. These two cases offer a particularly good contrast, since the experimental data are for a nearly insulated condition over the porous region, whereas the numerical result involves a larger heat transfer rate as well as different mass transfer rate and freestream Mach number.

The function $\chi(x/L)$ is exactly the ratio $(T_w - T_{aw}) / (T_L - T_{aw})$ downstream of a cooled (or heated) region of uniform temperature but with no mass transfer. To complete this discussion, Fig. 9 is included, showing an interesting comparison that so far as the author is aware has not appeared elsewhere. Durgin²⁹ previously has shown the good agreement between his data and χ , but at that time the numerical calculations of Ref. 5 were not available. Since tabulated results were not at hand when Fig. 9 was drawn

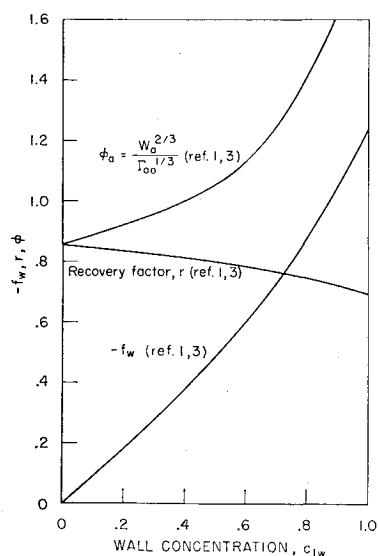


Fig. 15 Recovery factor r , function ϕ_a , and dimensionless mass transfer rate as functions of "wall concentration" for air injection

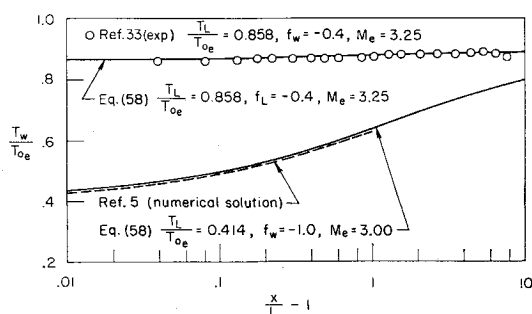


Fig. 16 Comparison of approximate results with numerical solution and experimental data for upstream cooling problem with air injection

there are likely some inaccuracies arising from reading curves, but the obvious agreement between the three sources certainly is true. The approximate result (i.e., χ) also can be obtained from Lighthill's formula,²³ from the work of Rubesin,³⁴ and of course from Hill's investigations.

Concluding Remarks

There is little question that the present results, based in large part on ideas due to Thwaites and Rott and Crabtree, are useful for helium and air injection. Extension to other situations seems largely a matter of obtaining the appropriate numerical solutions, at least for $dp/dx \neq 0$. Problems in which $dp/dx \approx 0$ lead to additional difficulties of two sorts: the momentum equation has added term, and the momentum thickness can be zero or negative.

It must be emphasized that the calculations depend on certain assumptions that can be justified only by the apparent correctness of the approximate results. The method leads only to formulas useful in calculating practically interesting quantities under relatively general conditions. This "recipe" is not to be confused with a "theory" of the boundary layer.

Appendix

Consider the integral equation for g :

$$F(x) = \int_a^x \left[\int_{\xi}^x h(x_1) dx_1 \right]^{-\beta} \frac{dg}{d\xi} d\xi$$

where $F(x) = 0$ for $x < a$ and h is a known function. Multiply by

$$\left[\int_x^y h(x_1) dx_1 \right]^{\beta-1} h(x) dx$$

and integrate over $a \leq x \leq y$:

$$\int_a^y \left[\int_x^y h dx_1 \right]^{\beta-1} h \Gamma dx = \int_{x=a}^{x=y} \left[\int_x^y h dx_1 \right]^{\beta-1} \times h dx \int_{\xi=a}^{\xi=y} \left[\int_{\xi}^x h dx_1 \right]^{-\beta} \frac{dg}{d\xi} d\xi$$

Inversion of the order of integration on the right-hand side gives

$$\int_{\xi=a}^{\xi=y} \frac{dg}{d\xi} d\xi \int_{x=\xi}^{x=y} h(x) \left[\int_x^y h dx \right]^{\beta-1} \left[\int_{\xi}^x h dx_1 \right]^{-\beta} dx$$

Now set

$$z = \frac{\int_{\xi}^x h dx_1}{\int_{\xi}^y h dx_1}$$

so that

$$h(x) dx = dz \int_{\xi}^y h dx_1$$

$$\int_x^y h dx_1 = (1 - z) \int_{\xi}^y h dx_1$$

Making these substitutions in the integral over x gives

$$\int_0^1 z^{-\beta} (1 - z)^{\beta-1} dz = \frac{\pi}{\sin \beta \pi}$$

and the desired inversion follows directly:

$$g(x) - g(a-) = \frac{\sin \beta \pi}{\pi} \int_a^x \left[\int_{\xi}^x h dx_1 \right]^{\beta-1} h(\xi) \Gamma(\xi) d\xi$$

This is the solution to the given integral equation for any integrable function h . Tribus and Klein³⁸ previously have shown this to hold for $h \sim x^{\alpha}$.

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